**Tensors and stuff**

(note we’re using Einstein summation convention, implicitly; maybe see Appendix at bottom)

**2. Basis vectors**

Most of the time, when we have some surface, we want to be able to describe directions in order to deal with vectors on the surface. All we need to describe vectors are magnitudes and directions. Let α be unit vectors pointing in the direction tangent to the coordinate line uα.

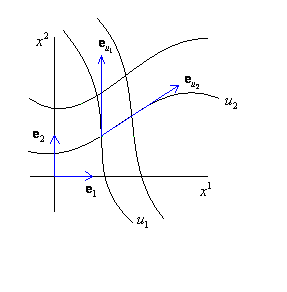
The covariant basis vectors are described as follows. Let d**r** be the vector (i.e. distance and direction) that points from uα → uα + duα, all other coordinates held constant, i.e., which goes along α and has magnitude drα. Then the covariant basis vector is given by (no implicit summation):



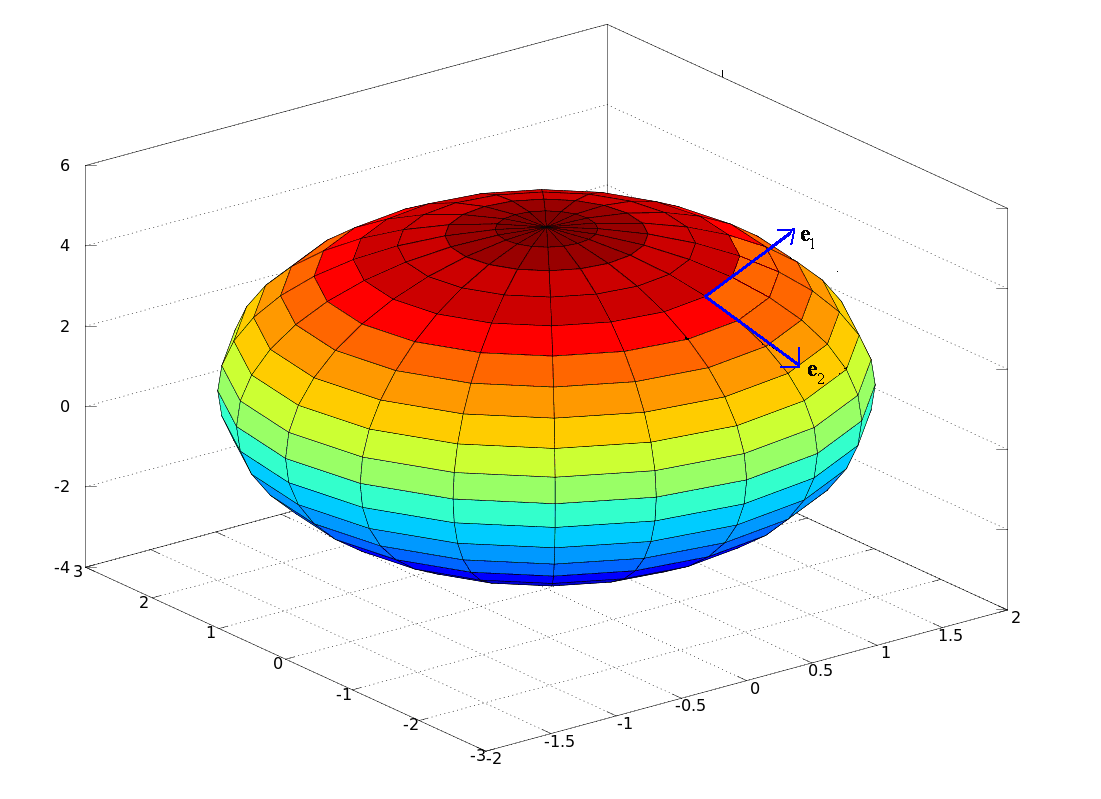
For instance a flat polar coordinate space would have:



This is illustrated more generally for a Cartesian coordinate system, and curvilinear coordinate system in flat space,



and again for some curved space coordinate system,



Note that we can define these vectors by standing within the system – no need to be outside. We simply vary the coordinate, measure the distance, and divide by the change in coordinate. It follows that for small displacments, an arbitrary displacement may be written as: d**r** = duα**e**α.

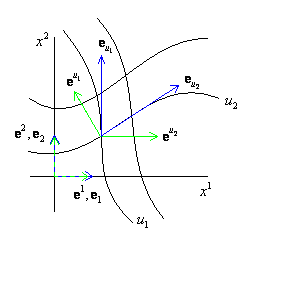
It should be clear from the examples above that this basis set is not necessarily orthonormal, or even orthogonal (different coordinate systems could result in a metric with off-diagonal elements). It is convenient therefore to introduce a second basis set called the contravariant basis vectors, **e**α, which are orthonormal to the covariant one [these are called one-forms in more rigorous treatments]. So let stand for unit vectors which point perpendicular to the surface of constant uα. Then the contravariant basis vectors are defined as follows. Let d**r** be the vector that goes perpendicular to the surface of constant uα, i.e., along , some distance dr. Then the coordinate uα will have changed by some amount duα. So divide duα by dr and multiply by ; this is the contravariant basis vector. Simply, it is the gradient of the constant uα surface (no implicit summation).



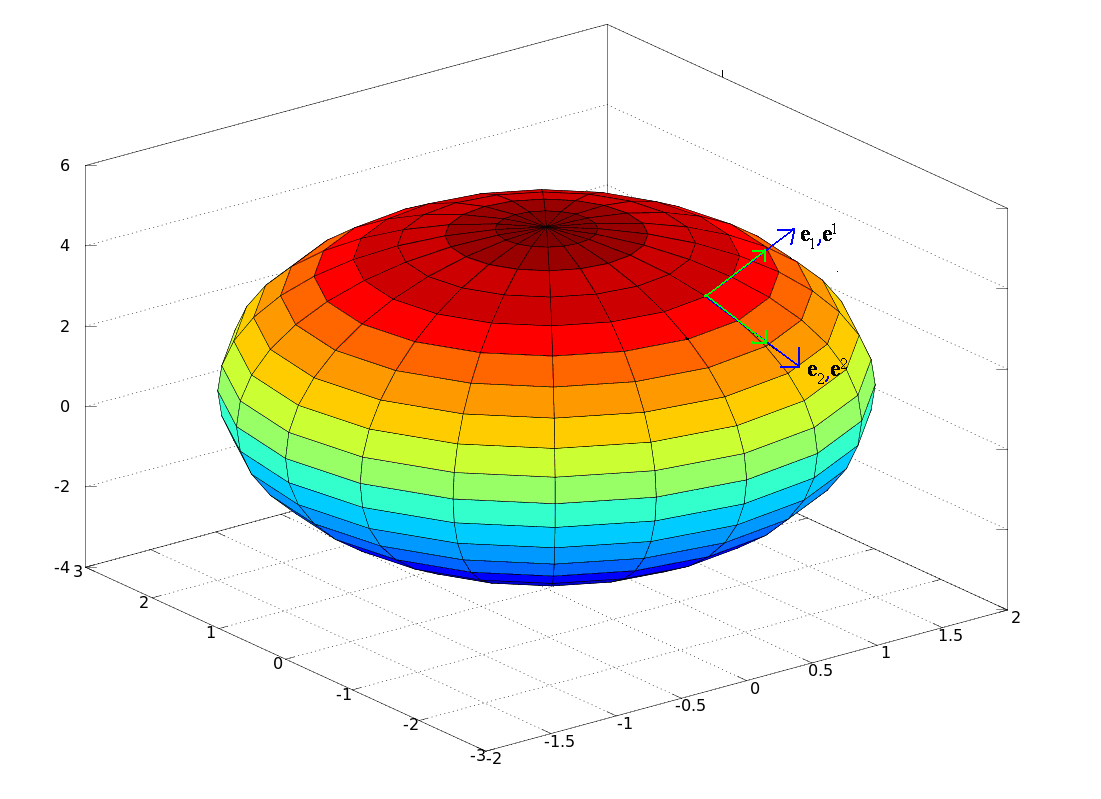
For instance a flat polar coordinate space would have:



These basis vectors are displayed below for some flat surface draped by Cartesian axes and curvilinear coordinates axes.



and below for some curved surface draped with orthogonal coordinate axes.



Note that in the last example, since the coordinate lines cross perpendicular to each other, **e**α and **e**α point in the same direction. This always happens for orthogonal coordinate systems. Importantly, we should observe that we can define these contravariant vectors entirely within the system, just as we could the covarient ones. We don’t have to stand outside. By virtue of their definitions, these bases are mutually orthonormal since:



So,



Likewise,



We can relate these guys to the metric. We know/define that:



But we also know that



It follows therefore that:



This can be taken as the definition of the dot product within our geometry. The ‘dual’ metric is defined to be the product between **e**α and **e**β. So we have:



Now let’s figure out the relationship between the covariant and contravariant bases, as well as the relationship between gαβ and gαβ. OK, since the covariant basis set is complete, we can expand any vector in a linear combination of covariant basis vectors. So let’s expand a covariant basis vector in a linear combination of contravariant ones (implicit Einstein summation notation).



So we can say:



And reversing the process, we can say:



But dotting both sides by **e**γ we have:



So summarizing we see that:



and,



So we see that the metric functions as a sort of raising/lowering operator which can raise or lower the index of a basis vector, i.e. can convert it between the contravariant and covariant bases.

The dot-product we were using above must be taken as being defined by the operation of the metric, and doesn’t necessarily correspond to anything like **A**·**B** = Abcosθ (indeed in special relativity this product can give us a negative number). Instead we simply have **A**·**B** = AμBμ = gμνAμBν. Even still, if we can embed our geometry within a higher dimensional Euclidean one, like this for instance,



Then it follows that we can write the basis vectors as:



And it seems we can profitably use the nice old dot product stuff too. I’ll parenthetically mention some analogies with Dirac notation in Quantum Mechanics. So we can write a resolution of identity,



because for instance,



And this is like how we can write 1 = Σα|α><α| in quantum mechanics. We can associate the covariant basis vectors, **e**a, to kets: |ea>, and the contravariant basis vectors, **e**a, to bras: <ea|. We just have to get used to the fact that <ea| and |ea> don’t point in the same direction, which is a little different than the Hilbert Space analogy from QM. But they do have non-zero overlap <ea|ea>. Moreover, |ea> is perpendicular to all <eb| such that b ≠ a. So likewise, one can visualize <ea| as just pointing in the direction perpendicular to all the |eb> where b ≠ a. And just like in QM, we can use the resolution of identity/identity tensor to express a vector in a particular basis. We just say:

 or 

If our metric is Euclidean, then we can interpret the dot product as **A**·**B** = ABcosθ, where θ is the angle between the two vectors. And this would provide a nice geometric way to get the contravariant components of a vector, if we know the covariant basis. Evidently aj = **a**·**e**j, which is just the dot product of **a** with **e**j. For instance, here’s a red vector expressed geometrically in a covariant and contravariant basis. Observe **e**1 is perpendicular to **e**2 and **e**2 is perpendicular to **e**1.

Chart

Description automatically generated

and though it doesn’t quite look like it, at all actually, a1 would just be **e**1·**a**, illustrated below,

Chart

Description automatically generated

Anyway,

**Example**

For the Cartesian system, show that the covariant basis vectors are the usual orthonormal basis set . Show that the contravariant basis vectors are the same. Determine the relationship between the contravariant and covariant basis vectors in the hemisphere geometry.

So in the Cartesian coordinates, the covariant basis vectors are **e**x, **e**y, **e**z which point in the x, y, z directions. From the metric,



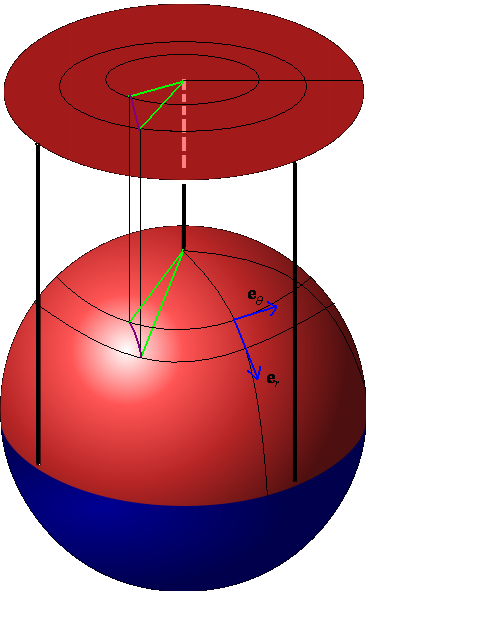
we see that **e**i·**e**j = δij and so the covariant basis vectors are all orthonormal. And so the covariant basis coincides with the usual Cartesian basis. The contravariant basis on the other hand is related to the covariant one according to:



So they are indeed the same.

**Example**

The covariant basis vectors (for the coordinate system chosen) for the hemisphere are: , illustrated below:



and you should observe that these vectors change direction as the point is varied. They also change magnitude, since they are normalized according to the metric which means,



The contravariant basis vectors are:



So:



So the contravariant basis vectors point in the same direction as the covariant ones, and this was to be expected because the coordinate system was orthogonal. Let’s do this another way. We can parameterize the points on the spherical surface via:



and in terms of r = Rsinθ, φ, we have:



and so the basis vectors would be:



and so the metric would be (note we can the Euclidean dot product here):



Another possibilitiy is to just keep the spherical coordinates and write the metric in terms of those two variables.

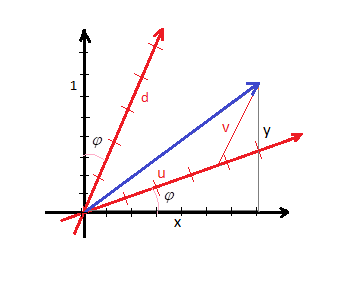


and so,



**Example**

Consider a 2D flat coordinate system. What are the covariant, contravariant basis vectors of the u,v system (in red), which is tilted inwards from the x-y system by angle φ? And let’s say that the coordinate system is stretched by a factor *d*. And what is its metric? BTW, this is similar to how time-space coordinates transform in special relativity, where φ is related to speed. Different metric than the one we’ll use here though.



The relationship between ‘unit’ vectors is (they don’t have unit magnitude):



i.e.,



The relationship between coordinates is:



and so,



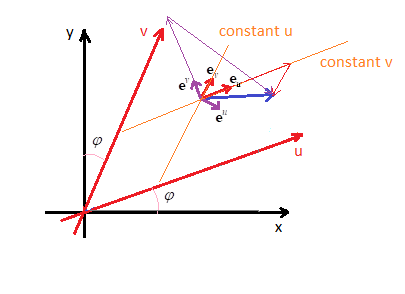
So the basis vectors of the (u,v) system are:



Can verify that these do form an orthonormal pair. In terms of the (u,v) system, the contravariant vectors are, for what it’s worth,



And we’ll note that eu,v, have components along each axis. And even though ev is perpendicular to eu, that doesn’t mean it doesn’t have a component along that direction. (don’t try to dot **e**u,v with **e**u,v in the u,v system w/o accounting for fact that u·v ≠ 0, etc.) Picture of a blue vector broken down into its components in each system. Note eu,v are parallel to the respective axes. And eu,v are perpendicular to constant u line and constant v line respectively. And if you examine it you can see eu,v have components along both eu,v.



Metric is, in the u,v system:



**Example**

Let’s make it a little more general. Let’s say that u is at an angle φu from the x-axis, and v is at an angle φv from the x axis. And then let’s stretch the two by du and dv. The relationship between basis vectors is:



i.e.,



The relationship between coordinates is:



which tells us that:



Taking transpose of both sides, to make it a little nicer, we have:



Now let’s get the covariant, contravariant basis vectors. So,



Can verify that these do form an orthonormal pair. For instance,



and,



Metric is, in the u,v system:



Note we can do the expansion (by ´, I mean u,v coordinate system)



So we can do the bra-ket Quantum Mechanics kind of thing, to get components of a vector, but have to use both coveriant and contravariant basis, basically. In other words, we can use a resolution of identity (of course implicit summation over basis vectors)

